SPHERICALLY SYMMETRIC STATIC CONFIGURATIONS OF UNIFORM DENSITY IN SPACETIMES WITH A NON-ZERO COSMOLOGICAL CONSTANT[1]

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Interior solutions of Einstein's equations with a non-zero cosmological constant are given for static and spherically symmetric configurations of uniform density. The metric tensor and pressure are determined for both positive and negative values of the cosmological constant. Limits on the outer radius of the interior solutions are established. It is shown that, contrary to the cases of the limits on the interior Schwarzschild and Reissner–Nordström solutions with a zero cosmological constant, these limits do not fully coincide with the conditions of embeddability of the optical reference geometry associated with the exterior (vacuum) Schwarzschild–de Sitter spacetime.

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1 Introduction

Recent observations give strong arguments for the presence of a non-zero repulsive cosmological constant, $\Lambda > 0$. The 'concordance' models of the observations of the fluctuations of the microwave cosmic background and the measurements of the Hubble constant favor the parabolic model of the universe, deserved by the inflationary paradigm [1, 2]. The parabolic model with a zero cosmological constant remains strongly disfavored in comparison with the parabolic model including a repulsive cosmological constant, according to an analysis of the new globular cluster dating and baryon abundance constrains [3]. Moreover, the data of the high-redshift supernovae indicate a negative deceleration parameter, giving a direct evidence for a repulsive cosmological constant [4].

The standard cosmological models with a non-zero cosmological constant were extensively discussed from the theoretical point of view by Tolman [5], and in connection to observational cosmological parameters in [6], and recently by Börner [7]. Also properties of the black-hole (and naked-singularity) spacetimes with a non-zero cosmological constant were discussed extensively (see, e.g., [8, 9, 10, 11]).

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Nevertheless, it is interesting to investigate static and spherically symmetric solutions of Einstein's equations with a non-zero cosmological constant representing relativistic stars. It was shown by Schwarzschild [12], and extensively discussed in [6], that in the case of a zero cosmological constant the spacetime structure of relativistic stars with a uniform density can be explicitly given in terms of elementary functions. The interior Schwarzschild–de Sitter spacetimes of uniform density (for a repulsive cosmological constant) were discussed under some simplifying assumptions in [5]. Here, it will be shown that the structure of relativistic stars of uniform density can be determined by elementary functions for both positive and negative values of the cosmological constant. It is useful to consider also negative values of the cosmological constant, because it was recognized recently that the anti-de Sitter spacetimes play an important role in the low-energy limit of the superstring theory [13]. Moreover, it enables us to give a complete discussion of relations between the restrictions on existence of the static configurations, and the limits of embeddability of the optical reference geometry [14], associated with the vacuum solutions of Einstein's equations with $\Lambda \neq 0$ describing the spacetime outside the static configurations.

In Section 2, Einstein's equations with a non-zero cosmological constant and the conservation law of energy-momentum tensor are used in the case of spherically symmetric spacetimes to give the equations of structure of spherically symmetric and static configurations representing relativistic stars. In Section 3, the equations of structure are explicitly integrated for the configurations of uniform density, and the pressure and metric tensor inside of these configurations are given. The interior geometry with a non-zero cosmological constant is always smoothly matched to an appropriately chosen exterior vacuum Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter geometry with the same cosmological constant. Reality conditions on the existence of these static configurations, giving limits on their outer radius, are determined.

Concluding remarks, concerning relations of the interior Schwarzschild–de Sitter spacetime presented in this paper to the general Stephani solutions of Einstein's equations, and relations between the limits on the outer radius of the static configurations and the reality conditions of embeddability of optical reference geometry related to the external vacuum Schwarzschild–de Sitter (and Schwarzschild–anti-de Sitter) spacetimes, are presented in Section 4.

2 Equations of structure

In terms of the standard Schwarzschild coordinates (t, r, θ, ϕ), the line element of spherically symmetric, static spacetimes can be given in the form

$$ds^2 = -\mathrm{e}^{2\Phi} dt^2 + \mathrm{e}^{2\Psi} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

with just two unknown functions, $\Phi(r)$ and $\Psi(r)$. (In the following, we shall use the geometric system of units with $c = G = 1$.) To high precision, the matter inside any star can be assumed to be a perfect fluid with the stress-energy tensor in the form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu} \tag{2}$$

where, in rest-frame of the fluid, $\rho = \rho(r)$ is the density of mass-energy, $p = p(r)$ is the isotropic pressure, $u^\mu = u^\mu(r)$ is the 4-velocity of the fluid, and $g^{\mu\nu}$ is the metric tensor of the spacetime. In a static star, each element of the fluid must remain at rest in the static coordinate system. Therefore,

$$u^r = \frac{dr}{d\tau} = 0 \quad u^\theta = \frac{d\theta}{d\tau} = 0 \quad u^\phi = \frac{d\phi}{d\tau} = 0 \tag{3}$$

and from the normalization condition it follows that

$$u^t = \frac{dt}{d\tau} = \mathrm{e}^{-\Phi}. \tag{4}$$

The fluid have to be characterized by an equation of state $p = p(\rho)$, or, in an implicit form, by the number density in rest-frame of the fluid $n = n(r)$, and the functional dependence of ρ and p on n: $\rho = \rho(n)$, $p = p(n)$.

The structure of a relativistic star is then determined by Einstein's field equations, $G_{\mu\nu} \equiv R_{\mu\nu} - Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$, and by the law of local energy-momentum conservation, $T^{\mu\nu}{}_{;\nu} = 0$. It is convenient to express the equations in terms of the components on the orthonormal tetrad of 4-vectors carried by the fluid elements:

$$\boldsymbol{e}_{(t)} = \frac{1}{\mathrm{e}^\Phi}\frac{\partial}{\partial t} \tag{5}$$

$$\boldsymbol{e}_{(r)} = \frac{1}{\mathrm{e}^\Psi}\frac{\partial}{\partial r} \tag{6}$$

$$\boldsymbol{e}_{(\theta)} = \frac{1}{r}\frac{\partial}{\partial\theta} \tag{7}$$

$$\boldsymbol{e}_{(\phi)} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \tag{8}$$

Projection of $T^{\mu\nu}{}_{;\nu} = 0$ orthogonal to u^μ (by the projection tensor $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$) gives the relevant equation

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \tag{9}$$

which is the equation of hydrostatic equilibrium describing the balance between gravitational force and pressure gradient.

There are two relevant structure equations following from the Einstein equations. These are determined by the $(t)(t)$ and $(r)(r)$ tetrad components of the field equations (the $(\theta)(\theta)$ and $(\phi)(\phi)$ components give dependent equations).

First we shall discuss the $(t)(t)$ component:

$$G_{(t)(t)} = \frac{1}{r^2} - \frac{\mathrm{e}^{-2\Psi}}{r^2} - \frac{1}{r}\frac{d}{dr}\,\mathrm{e}^{-\Psi} - \Lambda = 8\pi\rho. \tag{10}$$

We can transfer it into the form

$$\frac{d}{dr}\left[r\left(1 - \mathrm{e}^{-2\Psi}\right) - \frac{1}{3}\Lambda r^3\right] = \frac{d}{dr}2m(r) \tag{11}$$

where

$$m(r) = \int_0^r 4\pi r^2 \rho \, dr. \tag{12}$$

The integration constant in (12) is chosen to be $m(0) = 0$, because it means the spacetime geometry smooth at the origin (see [6]). Then we find the relation

$$\mathrm{e}^{2\Psi} = \left[1 - \frac{2m(r)}{r} - \frac{1}{3}\Lambda r^2\right]^{-1}. \tag{13}$$

The $(r)(r)$ component of the field equations reads:

$$G_{(r)(r)} = -\frac{1}{r^2} + \frac{\mathrm{e}^{-2\Psi}}{r^2} + \frac{2\mathrm{e}^{-2\Psi}}{r^2}\frac{d\Phi}{dr} + \Lambda = 8\pi p. \tag{14}$$

Using (13), we obtain the relation

$$\frac{d\Phi}{dr} = \frac{m(r) - \frac{1}{3}\Lambda r^3 + 4\pi p r^3}{r\left[r - 2m(r) - \frac{1}{3}\Lambda r^3\right]} \tag{15}$$

which enables us to put the equation of hydrostatic equilibrium (9) into the Tolman–Oppenheimer–Volkoff (TOV) form modified by the presence of a non-zero cosmological constant:

$$\frac{dp}{dr} = -(\rho + p)\frac{\left[m(r) - \frac{1}{3}\Lambda r^3 + 4\pi p r^3\right]}{r\left[r - 2m(r) - \frac{1}{3}\Lambda r^3\right]}. \tag{16}$$

For realistic equations of state, the equations of stellar structure can be integrated only numerically. However, the equations of structure can be integrated analytically for some idealized and ad hoc equations of state. We shall consider in the next section one of the most useful analytic solutions.

3 Relativistic configurations of uniform density

Relativistic model of a star of uniform density

$$\rho = \text{const for all } p \tag{17}$$

has been analytically integrated and discussed in the case of a zero cosmological constant [6, 12]. Here, it will be shown that the equations of structure can be analytically integrated even in the case of a non-zero cosmological constant. We shall discuss the cases of both repulsive, $\Lambda > 0$, and attractive, $\Lambda < 0$, cosmological constant.

Recall that in the case of a relativistic star with $\rho =$ const, it is not necessary to use the unrealistic notion of an 'incompressible fluid'. One can think of the fluids with pressure growing as radius decreases, having a composition that varies from one radius to another, and being 'hand-tailored' [6].

Assuming $\rho = \text{const}$, we can integrate the structure equations analytically. First, we obtain from the mass formula (12) that

$$m(r) = \frac{4\pi}{3}\rho r^3. \tag{18}$$

At the surface of the star ($r = R$), we obtain the total mass of the star

$$M = m(R) = \frac{4\pi}{3}\rho R^3. \tag{19}$$

Now, we can easily find the radial component of the metric tensor:

$$\mathrm{e}^{2\Psi(r)} = \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} \tag{20}$$

where we have introduced a new parameter by the relation

$$\frac{1}{\alpha^2} = \frac{1}{3}(8\pi\rho + \Lambda). \tag{21}$$

At the surface of the star, there is

$$\mathrm{e}^{2\Psi(R)} = \left(1 - \frac{R^2}{\alpha^2}\right)^{-1} = \left(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2\right)^{-1} \tag{22}$$

and we can see immediately that the radial metric coefficient of the interior spacetime is smoothly matched to the corresponding metric coefficient of the exterior Schwarzschild–de Sitter (or Schwarzschild–anti-de Sitter) spacetime of the mass parameter $M = m(R)$.

If $\rho = \text{const}$, the modified TOV equation (16) reduces into the form

$$\frac{dp}{(p + \rho)\,(3p + \rho - \Lambda/4\pi)} = -\frac{4\pi}{3}\,\frac{r\,dr}{(1 - r^2/\alpha^2)} \tag{23}$$

which have to be integrated from the surface of the star ($r = R$), where $p(R) = 0$, down to the center of the star at $r = 0$.

For a non-zero cosmological constant we find the pressure at a radius r to be given by the relation

$$p(r) = \frac{\rho\,(\rho - \Lambda/4\pi)\left[\left(1 - r^2/\alpha^2\right)^{1/2} - \left(1 - R^2/\alpha^2\right)^{1/2}\right]}{3\rho\,(1 - R^2/\alpha^2)^{1/2} - (\rho - \Lambda/4\pi)\,(1 - r^2/\alpha^2)^{1/2}}, \tag{24}$$

and for $\Lambda = 0$ this relation reduces to the well known formula given, e.g., in [6]. The maximum pressure is at the center of the star, where

$$p_{\mathrm{c}} = p(r = 0) = \frac{\rho\,(\rho - \Lambda/4\pi)\left[1 - \left(1 - R^2/\alpha^2\right)^{1/2}\right]}{3\rho\,(1 - R^2/\alpha^2)^{1/2} - (\rho - \Lambda/4\pi)}. \tag{25}$$

For fixed energy-density, ρ, and cosmological constant, Λ, the central pressure increases monotonously as the outer radius, R, increases, and also the mass, M, and the 'strength

of gravity', $2M/R = \frac{8}{3}\pi\rho R^2$, increase. (Naturally, as more matter is added to the star, a greater pressure is required to support it.)

The pressure at any relativistic star must be finite and positive. The restrictions

$$\rho - \frac{\Lambda}{4\pi} \geq 0 \tag{26}$$

$$3\rho\left(1 - \frac{R^2}{\alpha^2}\right)^{1/2} - \left(\rho - \frac{\Lambda}{4\pi}\right) \geq 0 \tag{27}$$

yield limits on the allowed values of outer radii R of the stars. The equality in (27) determines limiting configurations with a divergent central pressure. Substituting for ρ and α^2 from (19), and (21), respectively, and introducing new dimensionless cosmological and radius parameters by the relations

$$y = \frac{1}{3}\Lambda M^2 \tag{28}$$

$$x = \frac{R}{M} \tag{29}$$

the condition (27) can be transformed into the relation

$$[y - y_+(x)][y - y_-(x)] \leq 0 \tag{30}$$

where

$$y_\pm(x) \equiv \frac{2x - 9 \pm 3\,|2x - 3|}{2x^4}. \tag{31}$$

For the cosmological repulsion ($y > 0$) only the function

$$y_+(x) = \frac{4x - 9}{x^4} \tag{32}$$

is relevant (at $x \geq \frac{9}{4}$). (If $x = \frac{9}{4}$, we arrive at the well known limit of the interior Schwarzschild solutions [6].) However, the validity of the condition (30) is restricted to the region up to the maximum of $y_+(x)$, given by (32). It is located at $x_{\max} = 3$, where $y_{\max} = \frac{1}{27}$. At $x \geq x_{\max} = 3$, the relevant condition is (26) which determines a critical value of the cosmological constant for a given mass parameter M. In terms of the dimensionless parameters x and y, it implies the condition

$$y \leq y_{\mathrm{stat}} \equiv \frac{1}{x^3}. \tag{33}$$

Notice that for $y = y_{\mathrm{stat}}$, the outer radius of the star is located just at the so called static radius r_{s} of the corresponding external Schwarzschild–de Sitter spacetime. At $r = r_{\mathrm{s}}$, the gravitational attraction acting on a test particle is just compensated by the cosmological repulsion [9]. (At $r > r_{\mathrm{s}}$, the repulsion prevails, and a static configuration is possible only with a surface stress acting inwards. We shall not consider such a situation.)

For an attractive cosmological constant, $\Lambda < 0$, the relations (26) and (27) have to be satisfied again, but we obtain an other family of critical values of the cosmological

constant, given by the condition $1/\alpha^2 = 0$. In terms of the dimensionless parameters x and y, we arrive at

$$y_{\text{crit}} = -\frac{2}{x^3}; \tag{34}$$

in terms of the constant density ρ, the critical value of the cosmological constant is given by

$$\Lambda_{\text{crit}} = -8\pi\rho. \tag{35}$$

Now, we have to distinguish the cases $y > y_{\text{crit}}, y < y_{\text{crit}}$, and $y = y_{\text{crit}}$.

If $y > y_{\text{crit}}$ $(1/\alpha^2 > 0)$, the relations (30) and (31) are valid. Recall that at $x \geq \frac{3}{2}$, there is $y_-(x) = y_{\text{crit}}(x)$, while at $x \leq \frac{3}{2}$, there is $y_+(x) = y_{\text{crit}}(x)$. For $x = \frac{3}{2}, y_-(x) = y_+(x) = -\frac{16}{27}$. Therefore, in addition to $y > y_{\text{crit}} \equiv -2/x^3$, there must be satisfied the condition

$$-\frac{2}{x^3} \leq y \leq \frac{4x-9}{x^4} \tag{36}$$

at $x > 3/2$, and

$$\frac{4x-9}{x^4} \leq y \leq -\frac{2}{x^3} \tag{37}$$

at $x < 3/2$.

If $y < y_{\text{crit}}$ $(1/\alpha^2 < 0)$, the relation (30) has to be replaced by

$$[y - y_+(x)][y - y_-(x)] \geq 0. \tag{38}$$

In addition to $y < y_{\text{crit}} \equiv -2/x^3$, we obtain the conditions

$$y \leq -\frac{2}{x^3} \quad \text{or} \quad y \geq \frac{4x-9}{x^4} \tag{39}$$

at $x > 3/2$, and

$$y \geq -\frac{2}{x^3} \quad \text{or} \quad y \leq \frac{4x-9}{x^4} \tag{40}$$

at $x < 3/2$. It follows from the conditions (36)–(40) that for $\Lambda < 0$ the static configurations are allowed at radii satisfying the condition

$$y \leq \frac{4x-9}{x^4}. \tag{41}$$

In the special case when the outer radii of the static configurations are determined by the condition $y = y_{\text{crit}}(1/\alpha^2 = 0)$, the pressure is given by

$$p(r) = \frac{2\pi\rho^2\left(R^2 - r^2\right)}{1 - 2\pi\rho\left(R^2 - r^2\right)}. \tag{42}$$

In term of the dimensionless parameters, the central pressure of this special class of solutions is determined by the relation

$$p_{\rm c} = p(0) = \frac{3\rho}{2x-3} = -\frac{3\Lambda_{\rm crit}}{8\pi(2x-3)}. \tag{43}$$

Clearly, the special class of static configurations with $y = y_{\rm crit}$ is allowed for $x \geq 3/2$ only. Values of the cosmological parameter y must be restricted by the condition

$$-\frac{16}{27} \leq y < 0. \tag{44}$$

Finally, we determine the time component of the internal metric tensor, using the boundary condition of smooth matching of the internal solution onto the external time metric coefficient at $r = R$:

$$\mathrm{e}^{2\Phi(R)} = \left(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2\right). \tag{45}$$

The function $\Phi(r)$ can be found from Eq. (15) by using the relation for the pressure as a function of radius. If $1/\alpha^2 \neq 0$, we arrive at the expression

$$\begin{aligned}\mathrm{e}^{\Phi(r)} \;=\; & \frac{9M}{6M+\Lambda R^3}\left(1 - \frac{2M}{R} - \frac{1}{3}\Lambda R^2\right)^{1/2} - \\ & \frac{3M-\Lambda R^3}{6M+\Lambda R^3}\left(1 - \frac{2Mr^2}{R^3} - \frac{1}{3}\Lambda r^2\right)^{1/2}\end{aligned} \tag{46}$$

which holds at $r \leq R$ equally for both cases $y > y_{\rm crit}$ and $y < y_{\rm crit}$. At $r = R$ the relation (46) really reduces to (45). We can convince ourselves that the reality condition $\mathrm{e}^{\Phi(r)} \geq 0$ will be satisfied under exactly the same circumstances, as those derived above for the positivity and finiteness of the pressure function $p(r)$.

In the special case $y = y_{\rm crit}$ $(1/\alpha^2 = 0)$, we obtain

$$\mathrm{e}^{\Phi(r)} = 1 + \frac{3M}{2R}\left(\frac{r^2}{R^2} - 1\right). \tag{47}$$

Notice that now

$$\mathrm{e}^{\Psi(r)} = 1 \tag{48}$$

for all $r \leq R$, and the space sections $t = \text{const}$ have purely 3-dimensional Euclidean geometry. We can immediately see that also in this special situation the reality condition $\mathrm{e}^{\Phi(r)} \geq 0$ yields the condition $R \geq \frac{3}{2}M$, obtained from the properties of $p(r)$.

4 Concluding remarks

The solution of Einstein's equations presented above is a particular case of the solutions found by Stephani ([15]) as one of the metrics which can be embedded in a flat five-dimensional space, or equivalently, the Krasiński metrics representing spacetimes made

of 0(3)-symmetric three-dimensional spacelike hypersurfaces strung onto a timelike line orthogonal to them [16]. These solutions were derived under assumption that the source in the field equations is a perfect fluid. Following Krasiński [16], we can write the general geometry in the form

$$\begin{aligned} ds^2 &= -D^2(r,t)\,dt^2 + (1+Kr^2)^{-1}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \\ D(r,t) &\equiv r(A(t)\sin\theta\cos\phi + B(t)\sin\theta\sin\phi + C(t)\cos\theta) + \\ &\qquad E(t)\left(1+Kr^2\right)^{1/2} + s, \end{aligned} \tag{49}$$

with $s = 1$ or 0. The energy density ϵ and pressure $p_{\rm tot}$ (both including the contribution coming from the cosmological term) are then given by the relations

$$8\pi\epsilon = 8\pi\rho + \Lambda = -3K \tag{50}$$

$$8\pi p_{\rm tot} = 8\pi p - \Lambda = 3K\left(1 - \frac{2s}{3D(r,t)}\right). \tag{51}$$

Notice that for $A = B = C = s = 0$, $E = 1$, $3K = \Lambda$, we obtain the de Sitter solution. The interior Schwarzschild–de Sitter or Schwarzschild–anti-de Sitter solutions, given above, can be obtained from (49) by the following procedure: First we must redefine the time coordinate by

$$t_{\rm (K)} = \frac{3M}{R^3}\,\alpha^2\left(1 - \frac{R^2}{\alpha^2}\right)^{1/2} t \tag{52}$$

and then we must adjust the parameters of the Krasiński metric by the relations

$$\begin{aligned} &A(t) = B(t) = C(t) = 0 \quad s = 1 \\ &K = -\left(\frac{2M}{R^3} + \frac{\Lambda}{3}\right) = -\frac{\Lambda}{3}(8\pi\rho + \Lambda) \quad E = -\frac{(\rho - \Lambda/4\pi)}{3\rho(1+KR^2)^{1/2}}. \end{aligned} \tag{53}$$

With these parameters, the relation (51) for the pressure reduces correctly to the relation (25). Notice that the metric is conformally flat [16]; this fact can be helpful in investigations of its geodesics structure.

The optical reference geometry, introduced in [14], is very useful in attempts to understand the character of spherically symmetric spacetimes. Geodesics of the optical geometry exhibit some interesting physical properties – they coincide with the possible trajectories of light rays, massive particles require a speed-independent orthogonal thrust in order to move along them, and gyroscopes transported along them do not precess with respect to the direction of motion [17]. Moreover, the optical geometry appears to be very useful in the analysis of a variety of unusual processes that take place around compact objects [18, 19, 20].

Properties of the optical spaces associated with spherically symmetric spacetimes can be appropriately represented by embedding diagrams of their central planes into the 3-dimensional Euclidean space. The embedding allows an accurate treatment of some non-trivial effects [21]. However, the embedding is possible for a limited part of the optical space only. It is interesting that in the case of Schwarzschild spacetimes the limit

of embeddability of the optical geometry ([14]), $r = \frac{9}{4}M$, coincides with the minimum possible radius of a static configuration of matter of uniform density with fixed mass M ([6]). Similarly, static configurations of charged matter with the charge parameter $Q = M$ have the minimum radius $r = M$ ([22]), which coincides with the limit of embeddability of the optical space in the case of Reissner–Nordström spacetimes [21].

Kristiansson, Sonego and Abramowicz [21] therefore presented a conjecture that the minimum radius of embeddability of the optical geometry into the Euclidean space coincides with the minimum radius of a static configuration of given spacetime parameters.

It has been shown in [23] that the limits of embeddability for the optical geometry of the Schwarzschild spacetimes with a non-zero cosmological constant (both positive and negative) are given by the relation

$$y =\leq y_{\mathrm{emb}} \equiv \frac{4\tilde{x} - 9}{\tilde{x}^4} \tag{54}$$

where $\tilde{x} = r/M$. Clearly, if this condition determines a minimum radius of embeddability, it coincides with the minimum of corresponding static configuration of uniform density (see (31)), and the conjecture is confirmed. However, this is not the whole story, because the condition (54) determines also a maximum radius of embeddability for the spacetimes with a repulsive cosmological parameter $y < \frac{1}{27}$, and this differs from the maximum radius of the corresponding static configuration, given by (33).

For the spacetimes with an attractive cosmological constant, $y < 0$, there is no outer limit of embeddability of the optical geometry and the inner limit coincides with the limit on radius of static configurations of uniform density. However, there exists a special class ($1/\alpha^2 = 0$) of the internal solutions which corresponds to the outer limit of embeddability of the ordinary induced geometry on $t = \mathrm{const}$ hypersurfaces [23].

A more detailed discussion of the properties of the static configurations with an uniform density, including the geodesic structure of the internal spacetimes, will be discussed elsewhere [24]. But it is worth to mention the existence of stable circular photon orbits inside the configurations having radius $R < 3M$; surprisingly, the limiting radius is independent of the cosmological constant. The radius $r = 3M$, corresponding to the unstable circular photon orbits in the external Schwarzschild–de Sitter and Schwarzschild–anti-de Sitter spacetimes plays a crucial role in embeddings of the optical reference geometry associated with these spacetimes [23].

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